Rough sets and degree modifiers

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Overview

1. Background: perceptual meanings, degree modifiers, fuzzy sets
2. Rough sets
3. An example
4. Generalisation and discussion

Perceptual meanings

Perceptual meanings are central for understanding linguistic expressions referring to physical objects.

Staffan [Larsson, 2015] has considered understanding perceptual meanings in terms of classifiers.

Some classifiers can be treated compositionally, e.g., “upper right” can be understood as the conjunction of “upper” and “right”.

Not as much so with modifiers such as “far”.

Left-or-right game

Another feature of Staffan's approach is the learning, or interaction, aspect. In the left-or-right-game two agents, facing a framed surface, are negotiating the meaning of "left" and "right".

1. A places an object in the frame.
2. B focuses on the object and assigns it an individual label.
3. A says either "left" or "right".
4. B interprets A's utterance; whether B's understanding of the utterance is consistent with B's take of the situation.
5. If an inconsistency occurs, B happily learns from A; otherwise B says "okay".

The fuzzy approach

How to model degree modifiers such as “far”?

In classical set theory, an object is either a member of a given set or not, making it unsuitable for this task.

Fuzzy sets has been proposed as a means to model vague or gradual concepts, and elements of fuzzy sets has a *degree* of membership. [Zadeh, 1965]

Membership functions

In classical set theory, a characteristic function for a set X is a function $f_X : U \to \{0, 1\}$. such that $x \in X$ iff $f_X(x) = 1$.

In fuzzy set theory, the possible values are real numbers in the closed interval $[0, 1]$, so $f_X : U \to [0, 1]$. Hence fuzzy set theory is a generalisation of classical set theory.

One problem seems to be how to evaluate the degree of membership in applications.

Suppose that X is a fuzzy set aimed to represent the meaning of "right", and that $f_X(a) = .5$ and $f_X(b) = .8$. Arguably, b is *more* to the right than a, but is b *far* right? And is a to the right at all?

Rough sets

Rough sets provide a complementary perspective for reasoning about vagueness or under uncertainty. A basic assumption is that we associate some information with each object, and that objects may be indiscernible with respect to the available information. [Pawlak, 1982]

This means that vague concepts cannot be characterised in terms of precise information about their elements. On the rough set approach, vague concepts are described in terms of two precise concepts: the upper and lower approximations.

The lower approximation of a set A is the set of objects that is surely in A.

The upper approximation of a set A is the set of objects that might be in A.

Rough sets, cont'd

The basis of rough sets is the use of an equivalence relation to partition the universe of discourse.

Let U be a non-empty finite set, and let $\simeq$ be an equivalence relation on U.

Equivalence classes on U: $[x] = \{y \in U : x \simeq y\}$.

Definable sets are (unions of) equivalence classes.

Formally, the lower approximation of A: $A_* = \{x \in U : [x] \subseteq A\}$

The upper approximation of A: $A^* = \{x \in U : [x] \cap A \neq \emptyset\}$

Rough sets, cont'd

The approximations give rise to three pairwise disjoint regions:

The positive region for A is A_*.

The negative region for A is $\{x \in U : [x] \cap A = \emptyset\}$.

The boundary region is all the remaining elements of U.

If the boundary region is empty, A is an ordinary set.

Information systems

In the real world, the equivalence relation is likely to come from an information system: essentially a database or a table.

Let $S = \langle U, P \rangle$, where P is set of possible attributes of the objects in U.

With each attribute $p \in P$, there is an associated set V_p of possible values of p.

If Q is a finite subset of P, then an equivalence relation $\simeq_Q$ is induced by $x \simeq_Q y$ iff $p(x) = p(y)$ for all $p \in Q$.

A musical example

Five foot two, eyes of blue [...]
Has anybody seen my girl?

(Henderson–Lewis–Young 1925)

The singer is probably wishing for a situation like this:

name	length	eye colour
Hilary	5'2"	blue
Dana	5'2"	brown
Kim	5'4"	blue

Here, all singleton sets are *definable* from the attributes *length* and *eye colour*: $\{\text{Hilary}\} = [\text{Hilary}]$, $\{\text{Dana}\} = [\text{Dana}]$, $\{\text{Kim}\} = [\text{Kim}]$.

Example, cont'd

A somewhat less fortunate situation, still less complicated than the real world:

name	length	eye colour
Hilary	5'2"	blue
Dana	5'2"	blue
Kim	5'4"	blue

Given attributes *length* and *eye colour*, Hilary and Dana fall in the same equivalence class.

Let $A = \{\text{Hilary}\}$. Then $A^* = [\text{Hilary}] = \{\text{Hilary}, \text{Dana}\}$.

On the other hand $A_* = \emptyset$.

Example, cont'd

Five foot two, eyes of blue [...]
Has anybody seen my girl?
[...]

If you run into five foot two covered with furs
Diamond rings, all those things
Bet your life it isn't her

Example, cont'd

name	length	eye colour	covered with
Hilary	5'2"	blue	rags
Dana	5'2"	blue	furs
Kim	5'4"	blue	rags

With the additional attribute *covered with*, all singleton sets are again definable.

Left-or-right revisited

Suppose A and B have been playing the left-or-right game for some time. Suddenly, A places an object to the far right, saying "far right".

In a careful setup, B can learn to classify this object as "far right". Assuming there is some metric on the frame, say Cartesian coordinates, we could partition the frame in equivalence classes. If the representation is based on rough sets, B would classify other objects in the same equivalence class as "far right".

If further granulation is introduced through interaction ("to the extreme right", "somewhat to the right"), these further attributes could be added to B's system. With each additional piece of information, the equivalence relation (as well as the definable sets) is refined.

Questions

How are the equivalence classes determined?

Objects with identical x-coordinates are likely to be in the same equivalence class. This is probably too fine grained.

Is the representation of "far right" automatically contained in the representation of "right"?

What algebraic structure do we get? The literature is not clear on whether approximations commute/distribute over set union and intersection.

We might want to consider equivalence classes overlapping to some extent.

Membership functions revisited

Recall the membership functions introduced earlier. A rough set membership function can be understood as expressing a conditional probability.

$$f_A(x) = \Pr(A|[x]) = \frac{|A \cap [x]|}{|[x]|}$$

We can then express the approximations using this:

$$A_* = \{x \in U : \Pr(A|[x]) = 1\}$$

$$A^* = \{x \in U : \Pr(A|[x]) > 0\}$$

Probabilistic rough sets

Supposing $0 \leq \beta \leq \alpha \leq 1$:

$$A_* = \{x \in U : \Pr(A|[x]) \geq \alpha\}$$

$$A^* = \{x \in U : \Pr(A|[x]) \leq \beta\}$$

Concluding remarks and questions

Good model for learning from interaction? By adding new data, the equivalence relation changes.

Rough set membership isn't gradual. The ability to model vagueness comes from the introduction of the boundary region.

How do we give a meaningful interpretation of the parameters α and β? This is somewhat similar to the problem of interpreting the degree of membership of fuzzy sets.

Is it at all viable to use rough sets to model degree phenomena?

Or do we need to tailor each representation, by choosing the right metric/attributes/equivalence relation?

Concluding quote

> *An examination of the literature suggests a lack of systematic studies on the semantics of some of the fundamental notions of rough sets. This has led to inconsistent interpretations of the theory, misuses of the theory, and meaningless generalizations of the theory.*

[Yao, 2011], pp. 249–250.

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